Quantum Jet Bundles

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Introduction

- Long term goal: field theory via the variational bicomplex in the setting of NCG $\rightarrow$ Jet bundles
- Vector bundles in non-commutative geometry: use Serre-Swan theorem

$$\text{vector bundles } E \to M \iff \text{fgp } C^\infty(M)\text{-module } \mathcal{E} = \Gamma(E)$$
$$k\text{th jet bundle } J^kE \to M \iff \text{fgp } C^\infty(M)\text{-module } \mathcal{J}^k_{\mathcal{E}}$$

- Atiyah exact sequence of $C^\infty(M)$-modules:

$$0 \to \Omega^1 \otimes_{C^\infty(M)} \mathcal{E} \to \mathcal{J}^1_{\mathcal{E}} \to \mathcal{E} \to 0$$
$$\text{splittings } j^1_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{J}^1_{\mathcal{E}} \iff \text{connections on } \mathcal{E}$$

where the $C^\infty(M)$-module structure on $\mathcal{J}^1_{\mathcal{E}} \simeq \mathcal{E} \oplus \Omega^1 \otimes_{C^\infty(M)} \mathcal{E}$ given by

$$f.(s+\omega) = fs + \mathrm{d}f \otimes s + f\omega$$

In NCG ($C^\infty(M) \to A$):

- Construct A-module $\mathcal{J}^k_{\mathcal{E}}$ such that $j^k_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{J}^k_{\mathcal{E}}$ is a module map $\Rightarrow$ Connections ∇, braidings σ, Yang-Baxter equation

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Quantum differentials

- First-order differential calculus $(A, \Omega^1, \mathrm{d})$:
  - an A algebra over k
  - A-bimodule Ω^1
  - differential $\mathrm{d}\colon A \to \Omega^1$ obeying the Leibniz rule

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a(\mathrm{d}b)$$

  - $\Omega^1 = A\mathrm{d}A = \{a\mathrm{d}b | a, b \in A\}$
- Extended to DGA $\Omega = \bigoplus_n \Omega^n = T_A\Omega^1/\mathcal{I}$ for some ideal $\mathcal{I}$, with $\mathrm{d}^2 = 0$ and product $\wedge\colon \Omega^n \otimes_A \Omega^m \to \Omega^{n+m}$
  - In general $\omega \wedge \eta \neq (-1)^{|\omega||\eta|}\eta \wedge \omega$.
- Question: How to build higher derivatives?
  $\Rightarrow$ Connections

Quantum differential geometry

- Connections on an A-bimodule $\mathcal{E}$:

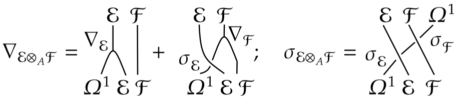
$$\nabla_{\mathcal{E}} : \mathcal{E} \to \Omega^1 \otimes_A \mathcal{E}, \qquad \sigma_{\mathcal{E}} : \mathcal{E} \otimes_A \Omega^1 \to \Omega^1 \otimes_A \mathcal{E}$$

  come with 'generalised braiding' $\sigma_{\mathcal{E}}$ for the Leibniz rules ($f \in A, s \in \mathcal{E}$)

$$\nabla_{\mathcal{E}}(fs) = \mathrm{d}f \otimes_A s + f(\nabla_{\mathcal{E}} s), \qquad \nabla_{\mathcal{E}}(sf) = (\nabla_{\mathcal{E}} s)f + \sigma_{\mathcal{E}}(s \otimes_A \mathrm{d}f).$$

- Tensor product connection: $\nabla_{\mathcal{E}}, \nabla_{\mathcal{F}}$ on $\mathcal{E}, \mathcal{F}$ induce a tensor product connection on $\mathcal{E} \otimes_A \mathcal{F}$ denoted by $\nabla_{\mathcal{E} \otimes_A \mathcal{F}}$.

$$\nabla_{\mathcal{E} \otimes_A \mathcal{F}} = \nabla_{\mathcal{E}} \otimes \mathrm{id} + (\sigma_{\mathcal{E}} \otimes \mathrm{id})(\mathrm{id} \otimes \nabla_{\mathcal{F}}), \qquad \sigma_{\mathcal{E} \otimes_A \mathcal{F}} = (\sigma_{\mathcal{E}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{\mathcal{F}})$$

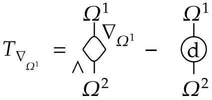


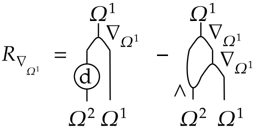
Quantum differential geometry

Case $\mathcal{E} = \Omega^1$:

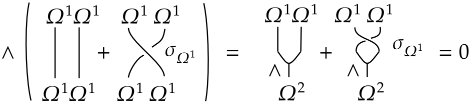
- $\nabla_{\Omega^1} : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$, $\sigma_{\Omega^1} : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$
- Torsion $T_{\nabla_{\Omega^1}} : \Omega^1 \to \Omega^2$ and curvature $R_{\nabla_{\Omega^1}} : \Omega^1 \to \Omega^2 \otimes_A \Omega^1$

$$T_{\nabla_{\Omega^1}} = \wedge\nabla_{\Omega^1} - \mathrm{d}, \qquad R_{\nabla_{\Omega^1}} = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla_{\Omega^1})\nabla_{\Omega^1}$$





- $T_{\nabla_{\Omega^1}} = 0$ implies $\wedge(\mathrm{id} + \sigma_{\Omega^1}) = 0$.



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First and second order jets

- Case of $M \times \mathbb{R} \to M$, i.e. $\mathcal{E} = C^\infty(M) = A$

Definition: The first and second order 'jet bimodules' $\mathcal{J}^1_A, \mathcal{J}^2_A$ and jet prolongation maps $j^1 \colon A \to \mathcal{J}^1_A$, $j^2 \colon A \to \mathcal{J}^2_A$

$$\mathcal{J}^1_A = A \oplus \Omega^1, \qquad \mathcal{J}^2_A = A \oplus \Omega^1 \oplus \Omega^2_S,$$
$$j^1(s) = s + \mathrm{d}s, \qquad j^2(s) = s + \mathrm{d}s + \nabla^2 s.$$

- Quantum symmetric forms $\Omega^2_S = \ker \wedge \subset \Omega^1 \otimes_A \Omega^1$
- 'Second-order derivative'

$$\nabla^2 := \nabla_{\Omega^1}\mathrm{d} \colon A \to \Omega^1 \otimes_A \Omega^1$$

$$\mathrm{d}s = \partial_i s\, \mathrm{d}x^i \qquad \Rightarrow \qquad \nabla^2 s = \partial_i \partial_j s\, \mathrm{d}x^i \otimes \mathrm{d}x^j + \cdots$$

- 'Second-order Leibniz rule' for $s, r \in A$

$$\nabla^2(sr) = (\nabla^2 s)r + \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \sigma_{\Omega^1} \Big] \mathrm{d}s \otimes \mathrm{d}r + s\nabla^2 r$$

where $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}; \sigma_{\Omega^1}\right] = \mathrm{id} + \sigma_{\Omega^1} : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ is a 'braided binomial'.

First and second order jets

- $\mathcal{J}_A^1 = A \oplus \Omega^1$, $j^1(s) = s + \mathrm{d}s$,
- $\mathcal{J}_A^2 = A \oplus \Omega^1 \oplus \Omega^2_S$, $j^2(s) = s + \mathrm{d}s + \nabla^2 s$.

Proposition

Given (A, Ω, d) and ∇_{Ω^1} torsion free, then $\mathcal{J}_A^1, \mathcal{J}_A^2$ are A-bimodules with the actions ($a \in A$)

$$a \bullet_1 (s + \omega_1) = as + (\mathrm{d}a)s + a\omega_1 \qquad (s + \omega_1) \bullet_1 a = sa + s\mathrm{d}a + \omega_1 a$$

on $(s + \omega_1) \in \mathcal{J}_A^1 = A \oplus \Omega^1$ and

$$a \bullet_2 (s + \omega_1 + \omega_2) = a \bullet_1 (s + \omega_1) + (\nabla^2 a)s + \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \sigma_{\Omega^1} \Big] \mathrm{d}a \otimes \omega_1 + a\omega_2$$

$$(s + \omega_1 + \omega_2) \bullet_2 a = (s + \omega_1) \bullet_1 a + s(\nabla^2 a) + \left[\begin{matrix} 2 \\ 1 \end{matrix}; \sigma_{\Omega^1}\right] \omega_1 \otimes \mathrm{d}a + \omega_2 a$$

on $(s + \omega_1 + \omega_2) \in \mathcal{J}_A^2 = A \oplus \Omega^1 \oplus \Omega^2_S$. The jet prolongations j^1, j^2 and obvious projection $\pi\colon \mathcal{J}_A^2 \to \mathcal{J}_A^1$, are bimodule maps.

Jets to all orders: ingredients

- For any k we take

$$\mathcal{J}^k_A = \bigoplus_{j=0}^{k} \Omega^j_S, \qquad j^k : A \to \mathcal{J}^k_A, \qquad j^k(s) = s + \mathrm{d}s + \sum_{j=2}^{k} \nabla^j s,$$

with the space of 'Quantum symmetric forms' and 'jth order derivative' given by

$$\Omega^j_S = \bigcap_i \ker \wedge_i \subset (\Omega^1)^{\otimes j}, \qquad \nabla^j = \nabla_{(\Omega^1)^{\otimes j}} \cdots \nabla_{\Omega^1} \mathrm{d} : A \to (\Omega^1)^{\otimes j}.$$

- For $k = 3$ and higher we need to impose extra conditions.

Jets to all orders: conditions

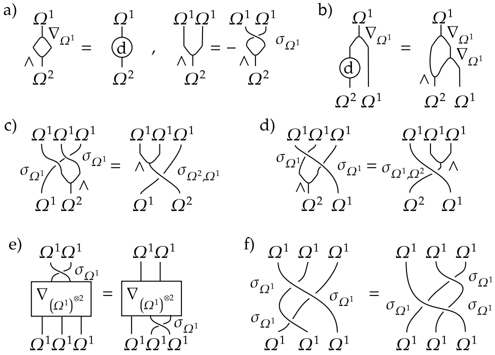
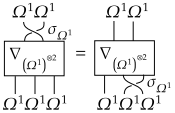


Figure: a) Torsion free, b) flat, c) $\wedge$-compatibility, d) extendability, e) Leibniz compatibility, f) braid relations.

Leibniz compatibility

- The ‘Leibniz compatibility’ condition

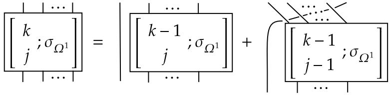


leads to the ‘kth-order Leibniz rule’

$$\nabla^k(sr) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}; \sigma_{\Omega^1} \Big] (\nabla^{k-j}s \otimes \nabla^j r).$$

- Braided binomials $\left[\begin{matrix} k \\ j \end{matrix}; \sigma\right] : (\Omega^1)^{\otimes k} \to (\Omega^1)^{\otimes k}$

$$\left[\begin{matrix} k \\ 0 \end{matrix}; \sigma_{\Omega^1}\right] = \mathrm{id} \quad ;$$



Jets to all orders

Theorem

Let ∇_{Ω^1} be a torsion free, flat, $\wedge$-compatible, extendable, Leibniz-compatible and with σ obeying the braid relations. Then $\mathcal{J}^k_A$, $j^k : A \to \mathcal{J}^k_A$

$$\mathcal{J}^k_A = \bigoplus_{j=0}^{k} \Omega^j_S, \qquad j^k(s) = s + \mathrm{d}s + \sum_{j=2}^{k} \nabla^j s,$$

form an A-bimodule and bimodule map with actions $\bullet_k$ given by

$$a \bullet_k \omega_j = j^{k-j}(a) \odot \omega_j, \qquad \omega_j \bullet_k a = \omega_j \odot j^{k-j}(a),$$

for $\omega_j \in \Omega^j_S$. Quotienting out Ω^k_S gives a bimodule surjection $\pi_k : \mathcal{J}^k_A \to \mathcal{J}^{k-1}_A$ such that $\pi_k \circ j^k = j^{k-1}$.

- Unital associative product on $\Omega_S = \bigoplus_{j=0}^{\infty} \Omega^j_S$: $\odot = \left[\begin{smallmatrix} k \\ j \end{smallmatrix}; \sigma_{\Omega^1}\right] : \Omega^{k-j}_S \otimes_A \Omega^j_S \to \Omega^n_S$
- Infinite jets: define $\mathcal{J}^\infty_A$ as the colimit of

$$\cdots \to \mathcal{J}^k_A \to \mathcal{J}^{k-1}_A \to \cdots \to \mathcal{J}^1_A \to \mathcal{J}^0_A = A.$$

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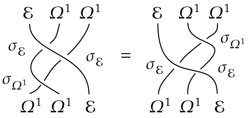
Concluding remarks and Outlook

Further remarks:

- Examples: $M_2(\mathbb{C}), S_3$, fuzzy $\mathbb{R}^3$, κ-Minkowski spacetime
- Vector bundle case $\mathcal{J}_\mathcal{E}^k = \mathcal{J}_A^k \otimes_A \mathcal{E}$
  - A-bimodule $\mathcal{E}$
  - Connection $\nabla_\mathcal{E}$ with $\sigma_\mathcal{E}$ satisfying 'coloured braid relations' (among others)



- $\Omega_S \subset T_A^{sh}\Omega^1 = (T_A\Omega^1, \odot)$ has the structure of a braided-Hopf algebra
- Approach with endofunctors (Flood, Mantegazza, Winther arXiv:2204.12401v1)

Next steps:

- Computation of further examples (fuzzy sphere, ...)
- Case with $R_{\nabla_{\Omega^1}} \neq 0$?
- Variational bicomplex

Thank You!